Mechanism with Efficient Steady State in Money Search Model.

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Abstract

This article considers an infinitely repeated economy with divisible fiat money. When money holding distribution is introduced in the model, it is shown that there exists a continuum of single-price equilibria indexed by the aggregate real-money balance, whenever one such equilibrium exists. There are two main existing models, (i.e. Lagos-Wright model and Shi model) to overcome with the problem of this real indeterminancy of steady state equilibria. This paper presents a new approach to prevent real indeterminancy of stationary equilibria. Allowing the existence of social planner in the model, this paper investigates the mechanism which implements efficient first-best outcome with non-degenerate money holding distribution in the steady state. Actual trade occurs at this steady state equilibrium. Although there are continuum of single-price equilibria, their differences are only nominal and so that my mechanism is free from real indeterminancy problem.

1 Introduction.

In our society, fiat money\(^1\) is used as an essential tool for transaction. Without fiat money, the exchange occurs only when there is a double coincidence of wants. The existence of fiat money allows seller with goods to give goods to buyer in return for certain amount of money. This usage of fiat money as a medium of exchange was first formally introduced in Kiyotaki & Wright (1989). In their model, agents are randomly matched to one another and they both simultaneously and independently decide to trade (either their goods or money) or not. This framework is widely used in many subsequent articles which I shall introduce next.

At first, monetary search model focused only on indivisible fiat money. Agents are assumed to produce and consume one unit of indivisible goods. It was shown by Kiyotaki & Wright (1998) that monetary equilibrium exists and economic welfare level is maximized when half of the agents in the economy owns money.

However, the trade method is restricted to one good for one unit of fiat money in those models, which implies that we can not analyze exchange rate between goods and fiat money. In order to analyze endogenous ”price” of good in equilibrium, economists introduced money holding distribution of agents in the model. The model with divisible fiat money was analyzed by Green & Zhou (1998). However, they found that there exists a continuum of equilibria with different price and welfare levels, if one such equilibrium exists. This is an unwanted problem from Macroeconomic policy making point of view. Moreover, Kamiya & Shimizu (2006) concluded that this real indeterminancy of steady state equilibria occurs not just in specific Green & Zhou model but also in wide class of models with money holding distribution.

There are two methods to overcome with this problem and brings unique steady state: Lagos-Wright model introduced by Lagos & Wright (2005) and Shi model introduced by Shi

\(^1\)Fiat money is defined as an money which itself does not produce any utility. It is often contrasted with commodity good, which itself brings utility.
In Lagos-Wright model, markets are divided into decentralized market (Day market) and centralized Warlasian market (Night market). Agents are randomly matched at the day market and become seller or buyer or neither. If one become a buyer or seller, his money holdings decreased or increased respectively. This difference in money holdings at the end of day market vanishes at the end of night market because at centralized night market, agents who were seller (resp. buyer) at decentralized day market may become a buyer (resp. seller). In this way, they have shown that there exists a unique stationary equilibrium in which all agents hold the same amount of money holdings.

In Shi’s model, there exists some large households containing continuum of agents. Each member of the household goes to random matching market and be a buyer or seller. Afterwards, they return to their household and share their goods and money so that after giving and receiving with in household, their money holdings become the same as the money holdings at the start of the day.

Although Lagos-Wright model and Shi model are widely used in preceding articles, their models have two problems. First, there is no evidence in our society which takes this form of trade. Second, even though there exists a unique stationary equilibria, their money holding distribution at steady state is degenerate so that the model assumes as if money is indivisible. There is no other model which brings to stationary equilibria with out degenerating money holding distribution.

In attempt to overcome with real indeterminancy of stationary equilibrium, Matsui & Shimizu (2005) thought that the underlying problem of real indeterminancy is exogenously given random match of agents. In order to reduce the friction due to randomness of match, they introduced ”marketplaces” with two identically equal sides and each agents can get to choose one marketplace they go to and one of the sides to take. It is assumed in their model that agents are randomly matched with agents who belongs to the opposite side of his marketplace. This way, we can reduce the probability of matches with no single coincidence of wants. In present article, I followed their idea.

In this article, I showed the existence of mechanism which brings steady state with non-degenerate money holding distribution. This mechanism implements the first-best allocation at steady state. Analyzing money search model using mechanism design framework is relatively new approach. It is first studied in Hu et. al (2009). Difference from their model to my model is that in Hu et. al, they used Lagos-Wright model as a general model setup but my model does not contain centralized market. Also, although I used marketplace introduced in Matsui & Shimizu (2005), my model and result is different from those of Matsui & Shimizu’s paper. Two main differences are characteristic of money holding distribution and concept of stationarity. Matsui & Shimizu focused on discrete money holding distribution, but my stationary money holding distribution is continuous. Second, Matsui & Shimizu showed efficiency in ‘evolutionary’ stable state but I showed efficiency in more widely used definition of stationarity.

The rest of paper is organized as follows. Section II introduce my framework and trading stages. Section III shows that this mechanism indeed implements the first best allocation. I conclude in section IV.

2 Model.

2.1 Basic Model.

Time is discrete $t = 0, 1, 2, \ldots$. I consider an infinite repetition of an economy. An economy consists of measure one of infinitely-lived agents and a social planner. There are two types of agents $j$ and $k$ with equal numbers who consume and supply. There are two types of perfectly divisible non-storable goods and a fiat money. Type $k$ agents produces type $k$ goods and consumes type $j \neq k$ goods. Since consumption goods are perishable, it can not be used as a
commodity money. In order to prevent gift-giving economy, it is assumed that agents cannot produce goods unless they expect to sell the goods with positive probability.

Let \( \eta_t \) denote agent’s money holdings at period \( t \). Thus for all \( t \), each agent is characterized by \( (\eta_t, i) \) where \( i \) denote his type and \( \eta_t \) the amount of true money holdings at \( t \). Agent’s types are publicly known but trading history and money holdings are private information.

Money holding distribution for the economy is somewhat tedious, so that I shall explain at the end of this section.

Agents’ discount rate is \( \delta \in (0, 1) \) and they are the same among all agents. Each agent has the same preference and technology. A utility of seller producing \( y \in \mathbb{R}_+ \) is \(-c(y)\) and utility of buyer consuming \( y \) is \( u(y) \), where \( c(0) = u(0) = 0 \), \( u' > 0 \), \( u'' < 0 \) and \( \lim_{y \to +0} u'(y) = +\infty \) and \( \lim_{y \to -\infty} = 0 \). For \( c \), \( c'(x) > 0 \), \( c''(x) < 0 \) for all \( x > 0 \). Assume \( c'(0) > 0 \). Assume there exits maximum amount of production, \( \bar{y} \in \mathbb{R}_+ \). This is a natural assumption because agents cannot put in infinite amount of labor. Thus, agents cannot consume infinite amount of goods. Also, \( c \) and \( u \) are strictly increasing and continuous with \( c \) convex and \( u \) concave and \( u - c \) strictly concave. Agent wants to maximize discounted sum of his utility \( \sum_{t=0}^{\infty} \delta^t u(y_t) \), where \( y_t \) denote the amount of consumption in period \( t \).

Assume there exist \( y_* > 0 \) such that \( y_* = \arg \max_{y} u(y) - c(y) \) with \( u(y_*) - c(y_*) > 0 \). Since \( u - c \) is strictly concave, the first best allocation \( y_* \) is uniquely determined. For notational simplicity, let \( u^* = u(y^*) \) and \( c^* = c(y^*) \).

As introduced in Matsui & Shimizu (2005), I shall introduce marketplaces. There are countably many marketplaces, indexed by \( z = 1, 2, 3 \ldots \) with two physically identical sides, \( X \) and \( Y \). Let the set of index of marketplace be \( N = \{ I, II, III, IV, \ldots \} \). At each period \( t \), each agents are randomly matched with one another agent in the opposite side within the same marketplace. Agents cannot go to more than one marketplaces at the same time.

Lastly, I shall explain money holding distribution of this model. In usual random matching model, it is assumed that borrowing money or holding negative amounts of money is not allowed because there are countably infinite number of agents so that you will not meet the person you met today. Hence all agents hold non-negative amount of money. Assume further that even though the economy does not increase the total amount of fiat money, agents are free to dispose their money holdings if they want to. However, agents are not allowed to give out fiat money to somebody else without any goods in return. (The same assumption can be said in a different way. Once you dispose fiat money to the ocean, no body can pick up and use it for trade.) Since there is production cost, it induces the upper bound on agents’ money holding which is endogenously determined. A very rich agent will choose not to be a seller even though he can increase his money holdings. This maximum level of money holdings is determined by the balance between marginal increase in value of money holding and the cost of production of corresponding amount of goods. Since it is assumed that \( c'(0) > 0 \), there exists endogenously chosen maximum level of money holdings.\(^2\) Let \( N > 0 \) denote the largest money holdings that agent wants to own. Then

\[
N = \min \{ \eta \in \mathbb{R}_+ \mid \delta( \text{Value of holding } \eta + p ) - \delta( \text{Value of holding } \eta ) < c^* \ \forall p \in \mathbb{R}_+ \}.
\]

For simplicity, normalize \( N = 1 \). Thus, rational utility maximizing agent has no incentive to hold more than \( \eta = 1 \) and agent with \( \eta = 1 \) will not choose to be a seller to increase his money holdings.

### 2.2 Trading Mechanism

In general mechanism designing problem, a mechanism is a rule that produces an outcome for any reported preferences. Following the framework used in Hu et.al. (2009), agent’s money

\(^2\)This is merely a conjecture but the existence of maximum level of money holding is justified in Zhou (1999) as well. I need to calculate the actual \( N \). For the rest of paper, I assume that my conjecture is correct.
holding can be thought of preference of this agent. Although all agents have the same utility function, agent’s current preferences are characterized by his value function (which I will explain at the end of section 2.3) and his value function depends only on his current money holding. Hence I conclude that declaring money holdings to social planner is almost as same as reporting his preference.

According to Mas-Collel, Whinston, & Green (1995), the basic framework of implementation in mechanism designing problem is as follows:
Step 1: Each agents sends a private message to social planner.
Step 2: Social planner creates a rule based on their messages so that he can achieve first best or optimal allocation in the economy.

Here, private message is assumed to be his money holdings. Each period consists of the following four stages.

- Stage 1: Declare their (not necessarily true) money holdings to social planner. (Later, I will be clear that it is the same as choice between seller or buyer.)
- Stage 2: Agents simultaneously choose a marketplace and one of its sides.
- Stage 3: In each marketplace, a random matching takes place between sides A and B.
- Stage 4: When type k agent and a type j ≠ k agent are matched,
  Step 1) they both know opponent’s true type. Social planner tells his partner’s (not necessarily true) declared money holdings.
  Step 2) Both simultaneously choose from {yes, no} to ‘planner’s rule’. If both announce ‘yes’ then trade takes place according to ‘planner’s rule’. If either buyer or seller announces no, they dispart without trade.

I am not allowing coalitional deviation. Since there are only two types of agents in this economy, whenever matching occurred between type k and type j agents, both wants what his opponent produces. It is natural to think that economic ‘runs well’ when agent with a lot of money holdings become a buyer and agent who are scant of money become a seller and increase their money holdings. In order to achieve this trading cycle, lets divide economy based on amount of money holdings.

Let economy of money holdings in [0, 1] be classified into four classes based on their money holdings: [0, 1/4), [1/4, 1/2), [1/2, 3/4), [3/4, 1].

Now I shall define the ‘planner’s rule’.

**Planner’s Rule.**
1) Agents with m ≤ 1/2 is a seller and ˜m ≥ 1/2 is a buyer.
2-i) When agents with m ∈ [0, 1/4) and ˜m ∈ [1/2, 3/4) meet, they trade goods y* in return for max{1/2 − m, ˜m − 1/2}.
2-ii) When agents with m ∈ [0, 1/4) and ˜m ∈ [3/4, 1] meet, they trade goods y* in return for max{1/2 − m, ˜m − 1/2}.
2-iii) When agents with m ∈ [1/4, 1/2) and ˜m ∈ [1/2, 3/4) meet, they trade goods y* in return for max{3/4 − m, ˜m − 1/4}.
2-iv) When agents with m ∈ [1/4, 1/2) and ˜m ∈ [3/4, 1] meet, they trade goods y* in return for max{3/4 − m, ˜m − 1/4}.
3) If you cannot follow the rule because of lack of money, your are forbidden to enter market again forever.

### 2.3 Strategies

Just like in many random matching search models, I shall assume that agents are using Markov strategies, i.e. their actions depend only on the current money holdings of the agent and not on his past endowment nor other’s endowments.
Define an agent’s strategy as follows.

**Definition.** Agents’ Markov strategy is defined to be a triple \( \sigma = (m, \lambda, s) \) where

i) \( m : [0, 1] \to [0, 1] \) (declared money holdings)

ii) \( \lambda : [0, 1] \times \{ j, k \} \to \mathbb{N} \times \{ X, Y \} \) (location strategy)

iii) \( s : [0, 1] \times [0, 1] \times [0, 1] \to \{ \text{yes, no} \} \) (response to planner’s rule)

Since agents are using Markov strategies, time subscript can be omitted because agents are facing the same problem every period. To avoid notational confusion, I usually used \( \eta \) to denote true money holdings and \( m \) for declared money holdings.

Define strategy \( \sigma^* \) as follows. From now, I will prove that \( \sigma^* \) is a Nash equilibrium.

**Definition.** Let \( \sigma^* \) be such that

\[
\begin{align*}
m(\eta) &= \begin{cases} 
0 & \text{if } \eta \in [0, \frac{1}{4}) \\
\frac{1}{3} & \text{if } \eta \in [\frac{1}{4}, \frac{1}{2}) \\
\frac{2}{3} & \text{if } \eta \in [\frac{1}{2}, \frac{3}{4}) \\
\frac{3}{4} & \text{if } \eta \in [\frac{3}{4}, 1] 
\end{cases} \\
\lambda(m, i) &= \begin{cases} 
(I, X) & \text{if } m \in [0, \frac{1}{2}) \text{ and type } j \\
(I, Y) & \text{if } m \in [\frac{1}{2}, 1] \text{ and type } k \\
(II, X) & \text{if } m \in [0, \frac{1}{2}) \text{ and type } k \\
(II, Y) & \text{if } m \in [\frac{1}{2}, 1] \text{ and type } j 
\end{cases}
\end{align*}
\]

and \( s(\eta, m, m') = \text{yes} \).

Intuition behind this strategy is simple. Since it is assumed to be autarky when he refuse social planner’s rule, agents must follow social planner’s rule in order to trade and therefore consume. Since buyer wants to buy paying lowest cost as possible and seller wants to sell at highest price as possible, there is no incentive to declare true money holdings to social planner. Since price function is determined by a maximum function, seller’s intention to sell at highest price as possible is carried over. Hence trade occurs at price \( \frac{1}{2} \), so that only agents who has more than or equal to \( \frac{1}{2} \) of money holdings wants to be a buyer because he will be punished forever if he can not pay \( \frac{1}{2} \).

I want to show that \( \sigma^* \) is a Nash equilibrium. In constructing Bellman equation of an agent, I need to fix all other’s actions. Since I am interested in Nash equilibrium, an agent’s value function can be written assuming all other agents are following \( \sigma^* \).

Even though there are only two types of agents randomly matched with another agents, there still exists frictions. For example, agent might be matched with the same type of agent, or the number of agents in side \( X \) is not equal to the number of agents in side \( Y \) so there are leftovers. Thus let \( r \in [0, 1] \) denote the probability of occurrence of match in which trade is possible. More precisely, fix an agent. Without loss of generality, assume he is type \( j \) who choose side \( W \) of market \( I \). Then

\[
r = \min \left\{ \frac{\text{number of agents in side } W \text{ of market } I}{\text{number of agents in opposite side}}, 1 \right\} \times 1_W
\]

where \( 1_W = \begin{cases} 
1 & \text{if } W = X \\
0 & \text{if } W = Y 
\end{cases} \). In words, \( r \) denote the probability that this agent meet someone times his partner is type \( k \).

Then according to this framework, the value function of a typical agent with money holding \( \eta \) can be calculated as follows.

Since he gets harsh punishment if he promise to pay more than his money holdings, I can assume that \( V(\eta) = -\infty \) if \( \eta < 0 \).
When $\eta \geq 0$,

$$V(\eta) = \max_{\text{yes}, \text{no}} \left\{ \delta V(\eta), \max_{\text{buyer, seller}} \left\{ \max_{m \in [0, \frac{1}{2}]} \left[ r[(-c(y))] + \delta V\left( \eta + \max_{m \in [\frac{1}{2}, 1]} \left\{ \frac{1}{4}, \frac{1}{2} - m \right\} \right] \right) \right\} \right\} + (1 - r) \delta V(\eta),$$

$$+ \delta V\left( \max_{m \in [\frac{1}{4}, 1]} \left[ r[u(y) + \delta V(\eta - \frac{1}{2})] + (1 - r) \delta V(\eta) \right] \right) \right\} \right\}$$

$$= \max_{\text{yes}, \text{no}} \left\{ \delta V(\eta), \max_{\text{buyer, seller}} \left\{ r[(-c^*) + \delta V(\eta + \frac{1}{2})] + (1 - r) \delta V(\eta), \right\} \right\}$$

$$\delta V(\eta), \max_{\text{buyer, seller}} \left\{ r[u^* + \delta V(\eta - \frac{1}{2})] + (1 - r) \delta V(\eta) \right\} \right\} \right\}$$

The choice between \{yes, no\} indicate the choice of either following planner’s rule or not and the choice between \{buyer, seller\} indicate the choice of declaring $m \geq \frac{1}{2}$ or $m < \frac{1}{2}$ to social planner. To be more precise, I need to include the choice of going to which marketplace and which side he is going to take within that marketplace. However, by construction of $\sigma^*$, when everybody else is following $\sigma^*$, there is no incentive to go to new market place nor choose ‘wrong’ side because it will bring him the same state as in choosing ’no’ to social planner’s rule. Since trade is not allowed unless social planner’s rule is carried over, there would only be trade of $y^*$ amount of goods.

Since agents are allowed to dispose money holdings, value function must be non-decreasing function of $\eta$. Furthermore, there is no incentive to hold infinite amount of money. Suppose to the contrary,

$$V(\eta + \frac{1}{2}) - V(\eta) \geq c^* \quad \forall \eta \in \mathbb{R}_+. \quad \cdots (\ast)$$

(\ast) only holds when $V$ is unbounded. But $V$ is bounded by maximum life time expected utility which is accomplished by agent consume at every time without producing. Since infinite amount of consumption is not possible because infinite production is not possible, it follows that (\ast) does not hold. Hence there is no incentive to actually hold infinite amount of money holdings. Thus, existence of $N$ given in previous section is justified. Again, to do more precise argument, I need to show actual value of $N$.

3 Stationary Equilibria.

3.1 Equilibria.

For notational simplicity, define $A = [0, \frac{1}{4}), B = [\frac{1}{4}, \frac{1}{2}), C = [\frac{1}{2}, \frac{3}{4}), D = [\frac{3}{4}, 1]$. Since coalitional deviation is not allowed, the only method for exchanging goods is by following social planner’s rule. Since social planner’s rule is divided based on four classes of money holdings, I conjecture that value function can be calculated based on these four classes also. Since all of agents within the same class are facing the same situation because they meet randomly and trade at the same price, it is natural to think that all agents within the same class has the same value function. This is shown in Lemma 1 below.
Lemma 1.  
- For all \( \eta, \eta' \in A \) with \( \eta \neq \eta' \), \( V(\eta) = V(\eta') \equiv V_A \).
- For all \( \eta, \eta' \in B \) with \( \eta \neq \eta' \), \( V(\eta) = V(\eta') \equiv V_B \).
- For all \( \eta, \eta' \in C \) with \( \eta \neq \eta' \), \( V(\eta) = V(\eta') \equiv V_C \).
- For all \( \eta, \eta' \in D \) with \( \eta \neq \eta' \), \( V(\eta) = V(\eta') \equiv V_D \).

Proof of Lemma 1. Since the only active equilibrium is the one in which social planner’s rule is carried over, the trade takes place with monetary transfer \( p = \frac{1}{2} \) and amount traded is \( y^* \). I will prove for \( \eta, \eta' \in A \) with \( \eta \neq \eta' \). Value function equality for other classes \( B, C, D \) can be shown similarly. From Bellman equation,

\[
V(\eta) = \max \{ r[-c^* + \delta V(\eta + \frac{1}{2})] + (1-r)\delta V(\eta), r[u^* + \delta V(\eta - \frac{1}{2})] + (1-r)\delta V(\eta), \delta V(\eta) \}.
\]

Thus, it needs to find

\[
\max \{ -c^* + \delta V(\eta + \frac{1}{2}), u^* + \delta V(\eta - \frac{1}{2}), \delta V(\eta) \}.
\]

Since \( \eta + \frac{1}{2} < \frac{3}{4} < 1 = N \) where recall that \( N \) is the endogenously chosen maximum amount of money holdings, it follows from definition of \( N \) that

\[
\delta V(\eta + \frac{1}{2}) - \delta V(\eta) \geq c^*
\]

holds. Thus, \(-c^* + \delta V(\eta + \frac{1}{2}) \geq \delta V(\eta)\). Since \( u^* + \delta V(\eta - \frac{1}{2}) = -\infty \) because \( \eta - \frac{1}{2} < 0 \), it follows that \( V(\eta) = r[-c^* + \delta V(\eta + \frac{1}{2})] + (1-r)\delta V(\eta) \). By rearranging,

\[
V(\eta) = \frac{r}{1-\delta + r\delta}(-c^*) + \frac{r\delta}{1-\delta + r\delta} V(\eta + \frac{1}{2}) \quad \cdots (*).
\]

Next, consider value for \( V(\eta + \frac{1}{2}) \). I will be shown in Lemma 4 that agent with \( \eta + \frac{1}{2} \) chooses to be a buyer. Thus, I do not want to get into much detail in this proof.

From Bellman equation,

\[
V(\eta + \frac{1}{2}) = \max \{ r[-c^* + \delta V(\eta + \frac{1}{2})] + (1-r)\delta V(\eta + \frac{1}{2}), r[u^* + \delta V(\eta)] + (1-r)\delta V(\eta + \frac{1}{2}), \delta V(\eta + \frac{1}{2}) \}.
\]

Thus, it needs to find

\[
\max \{ -c^* + \delta V(\eta + \frac{1}{2}), u^* + \delta V(\eta), \delta V(\eta + \frac{1}{2}) \}.
\]

Again since \( N = 1 \) denote endogenously chosen maximum money holdings, it follows that

\[
\delta V(\eta + 1) - \delta V(\eta) < c^*
\]

so that \(-c^* + \delta V(\eta + 1) < \delta V(\eta)\). Since \( \eta + \frac{1}{2} \leq \frac{3}{4} < 1 \), \( \delta V(\eta + \frac{1}{2}) - \delta V(\eta) \leq c^* < u^* \), it follows that \( u^* + \delta V(\eta) \geq \delta V(\eta + \frac{1}{2}) \). Thus, \( V(\eta + \frac{1}{2}) = r[u^* + \delta V(\eta)] + (1-r)\delta V(\eta + \frac{1}{2}) \). By rearranging,

\[
V(\eta + \frac{1}{2}) = \frac{r}{1-\delta + r\delta}(-c^*) + \frac{r\delta}{1-\delta + r\delta} V(\eta + \frac{1}{2})
\]

By substituting this result into \((*)\), I obtain

\[
V(\eta) = \frac{r}{1-\delta + r\delta}(-c^*) + \frac{r\delta}{1-\delta + r\delta} V(\eta + \frac{1}{2})
\]

\[
\iff V(\eta) = \frac{r}{1-\delta + r\delta}(-c^*) + \frac{r\delta}{1-\delta + r\delta} V(\eta)
\]

\[
\iff V(\eta) = \frac{(1-\delta + r\delta)r}{1-\delta}(-c^*) + \frac{r^2\delta}{1-\delta} u^*
\]
so that \( V(\eta) \) does not depend on \( \eta \). Hence \( V(\eta) = V(\eta') = V_A \) holds.

Now, I will focus on the incentive of choosing to be a buyer or seller more precisely. Again, Bellman equation can be written as follows:

\[
V_A = \max \{ r[-c^* + \delta V_C] + (1 - r)\delta V_A, r[u^* + \delta V(\eta - \frac{1}{2})] + (1 - r)\delta V_A, \delta V_A \}
\]

\[
V_B = \max \{ r[-c^* + \delta V_D] + (1 - r)\delta V_B, r[u^* + \delta V(\eta - \frac{1}{2})] + (1 - r)\delta V_B, \delta V_B \}
\]

for \( \eta \in C \),

\[
V(\eta) = \max \{ r[-c^* + \delta V(\eta + \frac{1}{2})] + (1 - r)\delta V_C(\eta), r[u^* + \delta V_A] + (1 - r)\delta V_C(\eta), \delta V_C \}
\]

for \( \eta \in D \),

\[
V(\eta) = \max \{ r[-c^* + \delta V_D(\eta + \frac{1}{2})] + (1 - r)\delta V_D(\eta), r[u^* + \delta V_B] + (1 - r)\delta V_D(\eta), \delta V_D \}
\]

Since agents are always allowed to dispose his money holdings, value function must be non-decreasing function of \( \eta \).

I conjecture that agents who has an ability to buy the product be a buyer and agents who can not afford to buy the product will sell the product so that he can be a buyer next period. I shall justify my conjecture by following Lemmas. Lemma 2 states that agents with \( \eta < \frac{1}{2} \) choose to be a seller.

**Lemma 2.** Agents in \( A \) and \( B \) choose to be a seller, i.e.

\[
V_A = r[-c^* + \delta V_C] + (1 - r)\delta V_A, \quad V_B = r[-c^* + \delta V_D] + (1 - r)\delta V_B.
\]

**Proof of Lemma 2.** It is obvious that agents choose not to be a buyer because \( V(\eta) = -\infty \) if \( \eta < 0 \). Since \( \delta V_C - \delta V_A \geq c^* \) from the definition of \( N = 1 \), it follows that \( -c^* + \delta V_C \geq \delta V_A \) so that

\[
V_A = \max \{ r[-c^* + \delta V_C] + (1 - r)\delta V_A, r[u^* + \delta V(\eta - \frac{1}{2})] + (1 - r)\delta V_A, \delta V_A \} = r[-c^* + \delta V_C] + (1 - r)\delta V_A.
\]

\( V_B \) can be checked similarly.

Now I proceed to state that agents with \( \eta \geq \frac{1}{2} \) choose to be a buyer. The proof of this statement contains two steps:

**Step 1)** There is no incentive to hold money \( \eta > 1 \).

**Step 2)** Since one can freely dispose money, there may be an incentive for agents in \( C \) to sell the product and obtain \( \frac{1}{2} \) and then throw away unwanted amount of money to still fit in to class \( D \). The same applies to \( \eta \in D \setminus \{1\} \). I show that there is no incentive for this behavior in Lemmas 2 and 3.

The first step is obvious from construction of the model. It is assumed that \( \eta = 1 \) is the maximum level of money holdings. Hence

\[
c^* > \delta V(\frac{3}{2}) - \delta V(1).
\]

Thus, for all \( \varepsilon > 0 \),

\[
c^* > \delta V(\frac{3}{2} - \varepsilon) - \delta V(1)
\]

so that there is no incentive to hold more than 1 unit of money.
Next I check the second step. When rich agents have the opportunity to sell product and throw away unwanted money, their value functions are given by the following bellman equation.

\[ V_C = \max \{ r[-c^* + \delta V_C] + (1-r)\delta V_C, r[u^* + \delta V_A] + (1-r)\delta V_C, \delta V_C \} \]
\[ V_D = \max \{ r[-c^* + \delta V_D] + (1-r)\delta V_D, r[u^* + \delta V_B] + (1-r)\delta V_D, \delta V_D \} \]

More precisely, I have to consider movement within class \( C \), i.e.

\[ V_C = \max \{ r[-c^* + \delta V_C] + (1-r)\delta V_C, r[-c^* + \delta V_D] + (1-r)\delta V_C, r[u^* + \delta V_A] + (1-r)\delta V_C, \delta V_C \} \]

but since \( V_C \leq V_D \), if I can show that \( r[-c^* + \delta V_D] + (1-r)\delta V_C \) is not a maximum value, then I can automatically know that \( r[-c^* + \delta V_C] + (1-r)\delta V_C \) is not a maximum value as well. I show in Lemmas 3 and 4 that in this situation, agents with \( \eta \geq \frac{1}{2} \) has no incentive to be a seller.

**Lemma 3.** Agents in \( D \) choose to be a buyer, i.e.

\[ V_D = r[u^* + \delta V_B] + (1-r)\delta V_D. \]

**Proof of Lemma 3.** Since \( r[-c^* + \delta V_D] + (1-r)\delta V_D < r\delta V_D + (1-r)\delta V_D = \delta V_D \), the Bellman equation is now becomes

\[ V_D = \max \{ r[-c^* + \delta V_D] + (1-r)\delta V_D, r[u^* + \delta V_B] + (1-r)\delta V_D \}. \]

Suppose \( V_D = r[u^* + \delta V_B] + (1-r)\delta V_D \) or in words, agents in \( D \) choose to be a buyer. Then by rearranging, we obtain

\[ V_B = \frac{(1-r)\delta}{\delta r} V_D - \frac{u^*}{\delta} \quad \cdots (1) \]

From Lemma 2,

\[ V_B = r[-c^* + \delta V_D] + (1-r)\delta V_B \quad \text{or} \quad V_B = \frac{r\delta}{1-\delta + r\delta} V_D + \frac{r}{1-\delta + r\delta} (-c^*). \]

Thus, by substituting \( V_B \) from (1), we obtain

\[ \frac{(1-\delta + r\delta)}{\delta r} V_D - \frac{u^*}{\delta} = \frac{r\delta}{1-\delta + r\delta} V_D + \frac{r}{1-\delta + r\delta} (-c^*) \]

\[ \iff V_D = \frac{r(1-\delta + r\delta)}{(1-\delta)(1-\delta + 2r\delta)} u^* - \frac{\delta r^2}{(1-\delta + 2r\delta)} (-c^*) \equiv V_D^*. \]

Now lets check that \( V_D^* \) is consistent with the original Bellman equation:

\[ V_D = \max \{ r[u^* + \delta V_B] + (1-r)\delta V_D, \delta V_D \}. \]

By substituting \( V_B \) from Lemma 2,

\[ \begin{align*}
\text{being buyer} & \quad \text{being autarky} \\
\frac{r[u^* + \delta V_B] + (1-r)\delta V_D}{\delta V_D} & \geq 0 \\
\iff u^* + \delta V_B - \delta V_D & \geq 0 \\
\iff u^* + \frac{r\delta}{1-\delta + r\delta} V_D & \geq 0 \\
\iff u^* + \frac{r}{1-\delta + r\delta} (-c^*) - \delta V_D & \geq 0
\end{align*} \]
so that by substituting $V_D^*$,

\[\iff u^* + \delta \frac{r \delta}{1 - \delta + r \delta} V_D^* + \frac{r}{1 - \delta + r \delta} (-c^*) - \delta V_D^* \geq 0\]
\[\iff (1 - \delta)u^* + r \delta (u^* - c^*) \geq 0\]

where the last equation is always satisfied for $\delta \in (0, 1)$ and $r \in [0, 1]$. Hence the value of agents in $D$ to be a buyer today is always greater than or equal to that of choosing autarky and strictly less than choose to be a seller. Hence $V_D = r[u^* + \delta V_B] + (1 - r)\delta V_D$. \hfill \qed

Lastly, I shall show in the same manner that agents in $C$ choose to be a buyer today.

**Lemma 4.** Agents in $C$ choose to be a buyer, i.e.

\[V_C = r[u^* + \delta V_A] + (1 - r)\delta V_C.\]

**Proof of Lemma 4.** Suppose $V_C = r[u^* + \delta V_A] + (1 - r)\delta V_C$ or in words, agents in $C$ choose to be a buyer. Then by rearranging, we obtain

\[V_A = \frac{(1 - \delta + r \delta)}{\delta r} V_C - \frac{u^*}{\delta}, \cdots (2)\]

From Lemma 2,

\[V_A = (1 - \delta + r \delta) \delta V_C - \frac{u^*}{\delta}, \cdots (2)\]

Thus, by substituting $V_A$ from (2), we obtain

\[\frac{(1 - \delta + r \delta)}{\delta r} V_C - \frac{u^*}{\delta} = \frac{r \delta}{1 - \delta + r \delta} V_C + \frac{r}{1 - \delta + r \delta} (-c^*).\]

\[\iff V_C = r(1 - \delta + r \delta) \delta V_C - \frac{u^*}{(1 - \delta + 2r \delta)} (-c^*) \equiv V_C^*.\]

Notice that $V_C^* = V_D^*$. 

Now let's check that $V_C^*$ is consistent with the original Bellman equation given $V_A, V_B, V_D$ in Lemmas 2 and 3. Recall the Bellman equation for $V_C$ was

\[V_C = \max\{r[-c^* + \delta V_D] + (1 - r)\delta V_C, r[u^* + \delta V_A] + (1 - r)\delta V_C, \delta V_C\}.\]

It needs to show that

\[
\begin{align*}
\text{being seller} & \quad \text{begin buyer} \\
\left(r[-c^* + \delta V_D] + (1 - r)\delta V_C \leq r[u^* + \delta V_A] + (1 - r)\delta V_C\right) & \quad \cdots (3) \\
\text{autarky} & \quad \text{begin buyer} \\
\delta V_C^* & \leq r[u^* + \delta V_A] + (1 - r)\delta V_C, \cdots (4)
\end{align*}
\]

always holds where $V_A = \frac{(1 - \delta + r \delta)}{\delta r} V_C - \frac{u^*}{\delta}$ from Lemma 2.

Since $r[-c^* + \delta V_D^*] + (1 - r)\delta V_C^* = r[-c^* + \delta V_C^*] + (1 - r)\delta V_C^* < \delta V_C^*$, I only need to check (4).

\[
\begin{align*}
\text{autarky} & \quad \text{begin buyer} \\
\delta V_C^* & \leq r[u^* + \delta V_A] + (1 - r)\delta V_C^* \\
\iff u^* + \delta V_A & \geq \delta V_C \\
\iff u^* + \delta \frac{r \delta}{1 - \delta + r \delta} V_C + \frac{r}{1 - \delta + r \delta} (-c^*) - \delta V_C & \geq 0
\end{align*}
\]
so that by substituting $V_C^*$,

$$\iff \ u^* + \delta \frac{r\delta}{1 - \delta + r\delta} V_C^* + \frac{r}{1 - \delta + r\delta} (\sigma^*) - \delta V_C^* \geq 0$$

$$\iff \ (1 - \delta) u^* + r\delta (u^* - c^*) \geq 0$$

where the last equation is always satisfied for $\delta \in (0, 1)$ and $r \in [0, 1]$. Hence the value of agents in $C$ to be a buyer today is always greater or equal to the value of choosing to be a seller or autarky. $V_C = r[u^* + \delta V_A] + (1 - r)\delta V_C$.

From Lemmas 2-4 above, I showed that agents who has an ability to pay $p = \frac{1}{2}$ will buy the product and agents who can not buy will sell goods so that he can be a buyer next period for any probability of trade.

This cycle can be confirmed by the following Proposition.

**Proposition 1.** There is no incentive to deviate from $\sigma^*$ when everyone is following $\sigma^*$.

**Proof of Proposition 1.**

There is no incentive to change sides within the same marketplace nor go to new marketplace because he is sure that he can not trade.

- $\eta \in [0, \frac{1}{4})$— If he become a seller, his optimal strategy is to sell at highest price as possible so that he will declare $0$. Since he will face the same payoff when declaring $\frac{1}{2}$, there is no incentive to deviate.

There is no incentive to be a buyer (i.e. $m \in (\frac{1}{4}, 1]$) because then he have to pay at least $\frac{1}{4}$ so that he would be punished forever almost sure.

- $\eta \in [\frac{1}{4}, \frac{1}{2})$— If he become a seller, his optimal strategy is to sell at highest price as possible so that he will declare $\frac{1}{4}$. Since he will face the same payoff when declaring $0$, there is no incentive to deviate.

There is no incentive to be a buyer because he might have to pay $\frac{1}{2}$ at max. Hence he will not be a buyer.

- $\eta \in [\frac{1}{2}, \frac{3}{4})$— If he become a buyer, his optimal strategy is to buy at lowest price as possible so that he will declare $\frac{1}{2}$. Since he will face the same payoff when declaring $\frac{3}{4}$, there is no incentive to deviate. There is no incentive to be a seller since he can be a buyer now as shown in Lemma 2.

- $\eta \in [\frac{3}{4}, 1]$— If he become a buyer, his optimal strategy is to buy at lowest price as possible so that he will declare $\frac{3}{4}$. Since he will face the same payoff when declaring $\frac{1}{2}$, there is no incentive to deviate. There is no incentive to be a seller since he can be a buyer now as shown in Lemma 3.

There is no incentive to say no to planner’s rule when opponent is saying yes because he can not trade otherwise. This establishes Remark 1.

3.2 Efficiency

In the preceding section, I showed that $\sigma^*$ is Nash equilibrium strategy. Now I will focus on the steady state.
For given $p \in [0, \frac{1}{2})$, define a money distribution such that for all $A \subset \mathbb{R}_+$, 

$$
\mu^p(A) = \begin{cases} 
0 & \text{if } p \not\in A, p + \frac{1}{2} \not\in A, \\
\frac{1}{2} & \text{if } p \in A, p + \frac{1}{2} \not\in A, \\
\frac{1}{2} & \text{if } p \not\in A, p + \frac{1}{2} \in A, \\
1 & \text{if } p \in A, p + \frac{1}{2} \in A 
\end{cases}
$$

or in shorthand, $\mu(\{p\}) = \frac{1}{2}, \mu(\{p + \frac{1}{2}\}) = \frac{1}{2}.$

Given $p \in [0, \frac{1}{2})$, money holding distribution $\mu^p$ divides population equally into two money holdings $\{p, p + \frac{1}{2}\}$.

In these money holding distribution, when every one is following $\sigma^*$, number of buyers equals the number of sellers in every market place so that $r = 1$.

Now I will focus on the steady state. The steady state I am referring to is the state in which
i) all agent with the same money holding and type act alike,
ii) agent uses time-independent strategy, and
iii) money holding distribution is time-independent.

More precisely, it is defined as follows.

**Definition.** A stationary state is a time invariant profile $(\mu, \sigma, V)$ such that
i) $\mu$ is stationary under $\sigma$,
ii) given $\mu, \sigma$ constitutes subgame perfect equilibrium, and
iii) given $\mu, V$ solves Bellman equation.

The following result shows that this stationary distribution is efficient.

**Proposition 2.** For any $p \in [0, \frac{1}{2})$, $\mu^p$ is an efficient active steady state where goods $y^*$ is traded for $\frac{1}{2}$ units of money.

**Proof of Proposition 2.** Let $p$ be any element in $[0, \frac{1}{2})$. Suppose money holding distribution is given by $\{p, \frac{1}{2} + p\}$ with $\frac{1}{2}$ of agents own $p$ and $\frac{1}{2}$ of agents own $\frac{1}{2} + p$.

From Proposition 1, $\sigma^*$ is a subgame perfect Nash equilibrium so that all agents follows $\sigma^*$.
Notice that when everyone is following $\sigma$, then trade occurs at price $\frac{1}{2}$.

This implies agents with $p$ become a seller and agent with $p + \frac{1}{2}$ become a buyer. Also, seller gives $y^*$ of goods in return for $\frac{1}{2}$ of money.

Then, in the next period, money holding distribution is given by $\{p, \frac{1}{2} + p\}$ with $\frac{1}{2}$ of agents own $p$ and $\frac{1}{2}$ of agents own $\frac{1}{2} + p$.

Thus, this distribution is a steady state and active.

Let $r$ denote the probability that $\eta \in A \cup B$ can trade and $\tilde{r}$ the probability that $\eta \in C \cup D$ can trade. From the definition of $\sigma^*$, $r = 1, \tilde{r} = 1$ holds because population is divided equally into two money holdings $\{p, p + \frac{1}{2}\}$.

Recall the value function in each classes are

$$
V_A = r[-c^* + \delta V_C] + (1 - r)\delta V_A \\
V_B = r[-c^* + \delta V_D] + (1 - r)\delta V_B \\
V_C = \tilde{r}[u^* + \delta V_A] + (1 - \tilde{r})\delta V_C \\
V_D = \tilde{r}[u^* + \delta V_B] + (1 - \tilde{r})\delta V_D.
$$

so that by inserting $r = 1 = \tilde{r}$ into the above equations and calculating, we obtain

$$
V_A^* = \frac{-c^* + \delta u^*}{1 - \delta^2}, \\
V_C^* = \frac{u^* + \delta (-c^*)}{1 - \delta^2}.
$$
Thus, 
\[
\text{Welfare} = \frac{1}{2} V_A^* + \frac{1}{2} V_C^* = \frac{1}{2} V_B^* + \frac{1}{2} V_C^* = \frac{1}{2(1 - \delta)} (u^* - c^*). 
\]

Since \(y^*\) is chosen such that \(u(y) - c(y)\) is maximized, it follows that welfare is maximized which implies that the economy at steady state is efficient.

Moreover, mechanism derived from social planner’s rule implements a first-best outcome because they are trading \(y^*\) amount of goods by following social planner’s rule.

Since \(p\) is chosen arbitrary from \(p \in [0, \frac{1}{2})\), there exists infinitely many steady state money holding distribution which is efficient, active and implements first-best outcome, but since the price and goods are fixed at \(\frac{1}{2y^*}\) per 1 good, these steady states are only nominally indeterminate.

4 Conclusion and Further studies.

This paper gives alternative method for acquiring stationary equilibrium distribution in the economy with money holding distribution. My result is distinguished from the other two existing approaches (i.e. Lagos-Wright model and Shi model) because steady state money holding distribution is not degenerate in my model. Also, even though there are infinite number of efficient stationary money holding distribution, they are only nominally different because trade occurs at the same price in all of these steady states. Hence, my result in this article gives one approach toward obtaining a solution for real indeterminancy problem.

Important restriction placed was that coalitional deviation from the social planner’s rule is not allowed. This gives a requirement that trade occurs at a certain way or else autarky. Further research will be needed considering coalitional deviation among randomly matched pair. Also, it would be better if I prove that \(\sigma^\ast\) is a subgame perfect equilibrium. However, in order to do so, I first need to define a subgame of this model which is quite complicated. In addition, it is an interesting question to study the existence of other equilibria in this model. Moreover, since payment rule in social planner’s rule is characterized by max function, it is almost as the same as seller’s-posting-price mechanism. Changing social planner’s rule might be a interesting extension of this model.

\footnote{It can be said that social planner’s rule is coalitional proof under some requirement on negotiation.}
References


